Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Sample Assessment Material

(Time: 1 hour 30 minutes) | Paper Reference **9FM0/01**

Further Mathematics

Advanced

Paper 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

1/1/1/1/

1. Solve the equation

$$6 \cosh 2x + 4 \sinh x = 7$$

giving your answers as exact logarithms.

(6)

(Total for Question 1 is 6 marks)

METHOD 1: using hyperbolic identities

notice how we can't solve the given equation without replacing the cosh double angle - ∴ using the hyperbolic version of the cos double angle:

$\cos 2x = 1 - 2\sin^2 x$ → using Osborne's rule - negating any explicit/implied product of sins; $\sin x \to \sinh x$ $(\sin^2 x)$ $\left(\tan^2 x = \frac{\sin^2 x}{\cos^2 x}\right)$

$$\cosh 2x = 1 + 2\sinh^2 x$$

and subbing into given equation

$$6(1+2\sinh^2 x) + 4\sinh x = 7$$

expand

$$6 + 12\sinh^2 x + 4\sinh x = 7$$

collect like terms

$$12\sinh^2 x + 4\sinh x - 1 = 0$$

↳ realising this as a quadratic in sinhx ∴ using substitution:

$$y = \sinh x$$

$$12y^2 + 4y - 1 = 0$$

use calc equation solver or quadratic formula

$$y = \frac{-4 \pm \sqrt{(4)^2 - 4(12)(-1)}}{2(12)}$$

$$= \frac{-4 \pm \sqrt{16+48}}{2(12)}$$

$$= \frac{-4 \pm \sqrt{64}}{24} = \frac{-4 \pm 8}{24}$$

... +ve: ...-ve:

$$y = \frac{4}{24} = 1/6 \qquad y = \frac{-12}{24} = -1/2$$

subbing into substitution $y = \sinh x$ ↳ solve for 'x':

$$\sinh x = 1/6 \qquad \sinh x = -1/2$$

$x = \text{arsinh}(1/6)$ $\qquad$ $x = \text{arsinh}(-1/2)$

WAY 1: using formula book definition for arsinhx

$$x = \ln\left(1/6 + \sqrt{(1/6)^2+1}\right) \qquad x = \ln\left(-1/2 + \sqrt{(-1/2)^2+1}\right)$$

$$= \ln\left(1/6 + \sqrt{1/36+1}\right) \qquad = \ln\left(-1/2 + \sqrt{1/4+1}\right)$$

$$= \ln\left(1/6 + \sqrt{37/36}\right) \qquad = \ln\left(-1/2 + \sqrt{5/4}\right)$$

$$= \ln\left(1/6 + \sqrt{37}/6\right) \qquad \boxed{= \ln\left(-1/2 + \sqrt{5}/2\right)}$$

$$\boxed{= \ln\left(\frac{1+\sqrt{37}}{6}\right)}$$

WAY 2: using exponential dfn for sinhx

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

...for $\sinh x = 1/6$

$$\frac{1}{2}(e^x - e^{-x}) = 1/6$$

$\div 1/2$ $\qquad$ $\div 1/2$

$$e^x - e^{-x} = 1/3$$

using index laws

$$e^x - \frac{1}{e^x} = 1/3$$

$\times e^x$ $\qquad$ $\times e^x$

$$e^{2x} - 1 = 1/3\, e^x$$

$$e^{2x} - 1/3\, e^x - 1 = 0$$

notice quadratic in e^x $\therefore$ using $y = e^x$ substition

$$y^2 - 1/3\, y - 1 = 0$$

calc equtn solver / quadratic formula

$$y = \frac{-(-1/3) \pm \sqrt{(-1/3)^2 - 4(1)(-1)}}{2}$$

$$y = \frac{1/3 \pm \sqrt{1/9 + 4}}{2}$$

common denominator inside root

$$y = \frac{1/3 \pm \sqrt{37/9}}{2}$$ taking 3 to denom. $\Rightarrow \dfrac{1 \pm \sqrt{37}}{6}$

using previous substitution:

$$e^x = \frac{1+\sqrt{37}}{6}$$

taking logs of both sides

$$\ln(e^x) = \ln\left(\frac{1 \pm \sqrt{37}}{6}\right)$$

can't take logs of -ves

$$\Rightarrow x = \ln\left(\frac{1+\sqrt{37}}{6}\right)$$

...now for $\sinh x = -1/2$

$$\frac{1}{2}(e^x - e^{-x}) = -1/2$$

$\div 1/2$ $\div 1/2$

$$e^x - e^{-x} = -1$$

using index laws

$$e^x - \frac{1}{e^x} = -1$$

$\times e^x$ $\times e^x$

$$e^{2x} - 1 = -e^x$$

taking to LHS:

$$e^{2x} + e^x - 1 = 0$$

↳ quadratic in e^x ∴ sub $y = e^x$ into above

$$y^2 + y - 1 = 0$$

↳ calc equation solver OR quadratic formula

$$y = \frac{(-1) \pm \sqrt{(1)^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

using $y = e^x$

$$e^x = \frac{-1 \pm \sqrt{5}}{2}$$

...+ve:

$$e^x = \frac{-1+\sqrt{5}}{2}$$

...-ve:

$$e^x = \frac{-1-\sqrt{5}}{2}$$

taking logs of both sides

$$x = \ln\left(\frac{-1+\sqrt{5}}{2}\right) \qquad x = \ln\left(\frac{-1-\sqrt{5}}{2}\right)$$

but can't take logs of -ves ∴ REJECT

$$\boxed{\Rightarrow x = \ln\left(\frac{1+\sqrt{37}}{6}\right), \ln\left(\frac{-1+\sqrt{5}}{2}\right)}$$

METHOD 2: using exponential dfns

using $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$

and $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

and subbing it into the given equation

$$6\left(\frac{1}{2}(e^{2x}+e^{-2x})\right) + 4\left(\frac{1}{2}(e^{x}-e^{-x})\right) = 7$$

expand

$$3e^{2x} + 3e^{-2x} + 2e^{x} - 2e^{-x} = 7$$

using index laws

$$3e^{2x} + \frac{3}{e^{2x}} + 2e^{x} - \frac{2}{e^{x}} = 7$$

$\times e^{2x}$

$$3e^{4x} + 3 + 2e^{3x} - 2e^{x} = 7e^{2x}$$

$$3e^{4x} + 2e^{3x} - 7e^{2x} - 2e^{x} + 3 = 0$$

noticing this is a quartic in $y = e^{x}$

$$3y^4 + 2y^3 - 7y^2 - 2y + 3 = 0$$

4 factorising this gives

$$(y^2+y-1)(3y^2-y-3) = 0$$

making each bracket equal to 0

$$y^2 + y - 1 = 0 \qquad 3y^2 - y - 3 = 0$$

4 using calc equtn solver

or quadratic formula

$$y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \qquad y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-3)}}{2(3)} = \frac{1 \pm \sqrt{37}}{6}$$

subbing into y substitution:

$$e^{x} = \frac{-1 \pm \sqrt{5}}{2} \qquad e^{x} = \frac{1 \pm \sqrt{37}}{6}$$

taking logs of both sides

$$x = \ln\left(\frac{-1 \pm \sqrt{5}}{2}\right) \text{ or } x = \ln\left(\frac{1 \pm \sqrt{37}}{6}\right)$$

can't take logs of -ves:

$$\boxed{\therefore x = \ln\left(\frac{1+\sqrt{37}}{6}\right), \ln\left(\frac{-1+\sqrt{5}}{2}\right)}$$

2. A company runs three theme parks, A (Aztec Adventureland), B (Babylonian Towers) and C (Carthaginian Kingdom).

It is known that park A makes a profit of £30 per visitor, park B makes a profit of £26 per visitor and park C makes a profit of £33 per visitor.

In 2017 the Aztec Adventureland park was upgraded, which took one year to carry out. During 2017

- park A had only 50% of the number of visitors it had in 2016
- park B had 25% more than the number of visitors it had in 2016
- park C had 15% more than the number of visitors it had in 2016

In total 1 350 000 people visited the three theme parks during 2017.

The company made a total profit from the parks of £39.15 million in 2016. The profits dropped by 1% for 2017.

Form and solve a matrix equation to find, to 2 significant figures, the number of visitors for each of the theme parks in 2016.

(8)

(Total for Question 2 is 8 marks)

realising the question is asking us to formulate and solve a matrix equation - hence need to identify the three linear equations needed to solve a matrix equation

... first defining variables :

let x = no. of visitors to park A in 2016

y = no. of visitors to park B in 2016

z = no. of visitors to park C in 2016

... next formulating the three linear equations using previously defined variables :

· 'profit equation' → 2016

$30x + 26y + 33z = 39.15 \times 10^6$ -①

· 'no. of visitors' → 2016 (% multiplier)

$0.5x + 1.25y + 1.15z = 1.35 \times 10^6$ -②

· 'profit equation' → 2017

↳ using profit equation (2016) and no. of visitors equation

$30(0.5x) + 26(1.25)y + 33(1.15)z = (39.15) \times 0.99 \times 10^6$

$= 15x + 32.5y + 37.95z = 38.7585 \times 10^6$ -③

...3 equations:

$30x + 26y + 33z = 39.15\times10^6$ -①

$0.5x + 1.25y + 1.15z = 1.35\times10^6$ -②

$15x + 32.5y + 37.95z = 38.7585\times10^6$ -③

...to solve, remembering general form for matrix equations: $M\underline{x} = \underline{y}$

$\Rightarrow \underline{x} = M^{-1}\underline{y}$

rewriting the coefficients of ①, ② and ③ as 'M', variables as the 'x' and integers as the 'y':

$$\begin{pmatrix} 30 & 26 & 33 \\ 0.5 & 1.25 & 1.15 \\ 15 & 32.5 & 37.95 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 39.15\times10^6 \\ 1.35\times10^6 \\ 38.7585\times10^6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 30 & 26 & 33 \\ 0.5 & 1.25 & 1.15 \\ 15 & 32.5 & 37.95 \end{pmatrix}^{-1}} \begin{pmatrix} 39.15\times10^6 \\ 1.35\times10^6 \\ 38.7585\times10^6 \end{pmatrix}$$

evaluating this using CLASSWIZ calc–
4.MATRIX - define MATRIX A - OPTN - MatA - x^{-1} button

$$= \begin{pmatrix} 0.0576 & 0.4916 & -0.065 \\ -9\times10^{-3} & 3.6871 & -0.103 \\ -0.014 & -3.351 & 0.1403 \end{pmatrix}\begin{pmatrix} 39.15\times10^6 \\ 1.25\times10^6 \\ 38.7585\times10^6 \end{pmatrix}$$

↳now evaluate using matrix multiplication – on CALC

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 400,331.7576 \\ 593,257.413 \\ 355,010.7434 \end{pmatrix}$$

$$= \begin{pmatrix} 400,000 \\ 590,000 \\ 360,000 \end{pmatrix} \text{ to 2 s.f}$$

$\therefore$ 400,000 visitors in park A in 2016
590,000 visitors in park B in 2016
360,000 visitors in park C in 2016

NOTE: could've also tried eliminating one of the variables in the three linear equations to get a simpler matrix equation but this is rather time-consuming and difficult when dealing with large decimals

3.

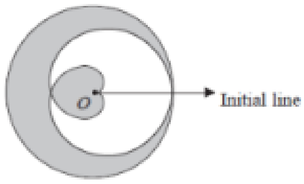


Figure 1

Figure 1 shows a sketch for the design of a logo. The logo is defined by the polar curve with equation

$$\boxed{r = \sin\left(\frac{\theta}{6}\right)} \qquad 0 \leq \theta \leq 6\pi$$

The inner closed section and outer closed section of the curve, shown shaded in Figure 1, are to be coloured the same colour. The remaining section is to be left clear.

(*a*) Use algebraic integration to find the area of the coloured sections of the logo.

(6)

A copy of this logo is to be painted on a white wall of a building such that the total width of the logo is 12 m.

Tins of coloured paint with an advertised minimum coverage area of 30 m^2 are to be used to paint the coloured sections of the logo onto the wall. Given that two coats of paint will be needed,

(*b*) find the minimum number of tins of this paint that should be bought to ensure that the coloured sections of the logo can be painted onto the wall.

(4)

(Total for Question 3 is 10 marks)

(a) notice 'use algebraic integration' requires us to use the formula for polar integration: $\int_\alpha^\beta r^2\,d\theta$ but apply it to a SPIRAL (i.e the design of the logo)

we are given the polar equation of the spiral - the 'r' - now need to trace the spiral to get the area of the required coloured sections

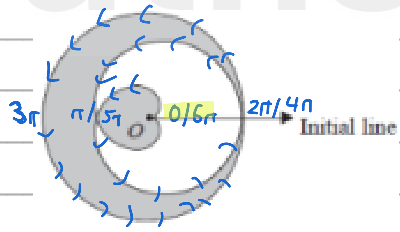


Figure 1

AND noticing symmetry - coloured components are :

between 3π and 2π

subtract non-coloured 5π and 4π

add coloured between π and 0

Subbing above into formula for polar integration:

area of coloured $= \cancel{2}\times\left(\cancel{\tfrac{1}{2}}\int_{2\pi}^{3\pi} r^2\,d\theta - \int_{4\pi}^{5\pi} r^2\,d\theta + \int_0^{\pi} r^2\,d\theta\right)$

first indefinitely integrate

$$\int \sin^2 \theta/6 \; d\theta$$

remembering that can't really integrate high trig powers ∴ use memorised REARRANGED cos double angle (one involving $\sin^2\theta$)

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

$$\int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos \theta/3\right) d\theta$$

now integrate using $\int \cos k\theta \, d\theta = \frac{1}{k}\sin k\theta + c$

$$= \tfrac{1}{2}\theta - \tfrac{3}{2}\sin\tfrac{\theta}{3} + c$$

now evaluate this integral at each of the limits:

$$\int_{2\pi}^{3\pi} r^2\,d\theta = \left[\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\tfrac{\theta}{3}\right]_{2\pi}^{3\pi} = \left\{\left[\tfrac{1}{2}(3\pi) - \tfrac{3}{2}\sin\left(\tfrac{\cancel{3}\pi}{\cancel{3}}\right)\right] - \left[\tfrac{1}{2}(2\pi) - \tfrac{3}{2}\sin\left(\tfrac{2\pi}{3}\right)\right]\right\}$$

$$= \frac{3\pi}{2} - 3/2(0) - \pi + 3/2\left(\sqrt{3}/2\right)$$

$$= \frac{3\pi}{2} + \frac{3\sqrt{3}}{4} - \pi$$

$$= 1/2\,\pi + \frac{3\sqrt{3}}{4}$$

$$\int_{4\pi}^{5\pi} r^2\,d\theta = \left[\tfrac{1}{2}\theta - \tfrac{3}{2}\sin\tfrac{\theta}{3}\right]_{4\pi}^{5\pi} = \left\{\tfrac{1}{2}(5\pi) - \tfrac{3}{2}\sin\left(\tfrac{5\pi}{3}\right)\right] - \left[\tfrac{1}{2}(4\pi) - \tfrac{3}{2}\sin\left(\tfrac{4\pi}{3}\right)\right]\right\}$$

$$= \frac{5}{2}\pi - \cancel{\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)} - 2\pi + \cancel{3/2\left(-\sqrt{3}/2\right)}$$

$$= 1/2\,\pi$$

$$\int_0^{\pi} r^2\,d\theta = \left[\tfrac{1}{2}\theta - \tfrac{3}{2}\sin\left(\tfrac{\theta}{3}\right)\right]_0^{\pi} = \left\{\left[\tfrac{1}{2}(\pi) - \tfrac{3}{2}\sin(\pi/3)\right] - \left[\tfrac{1}{2}(0) - \tfrac{3}{2}\sin(0)\right]\right\}$$

$$= \pi/2 - 3/2\left(\sqrt{3}/2\right)$$

$= \pi/2 - \frac{3\sqrt{3}}{4}$

$\therefore$ Area $= \frac{1}{2}\cancel{\pi} + \cancel{\frac{3\sqrt{3}}{4}} - \frac{1}{2}\cancel{\pi} + \pi/2 - \cancel{\frac{3\sqrt{3}}{4}}$

$\boxed{= \pi/2}$

(b) annotating Fig 1 with the given information

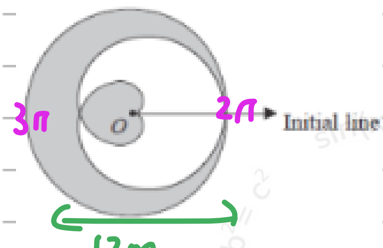


Figure 1

we know from (a) that area of the logo $= \pi/2$ $\therefore$ need to see how the area scales up so as to then find out the no. of tins of paint to be bought

↳ for this use given width = 12 and work out initial width

↳ to find this sub $\theta = 2\pi$ into 'r': and $\theta = 3\pi$,

$r = \sin\left(\frac{2\pi}{6}\right) = \sin(\pi/3)$ $\qquad$ $r = \sin\left(\frac{3\pi}{6}\right) = 1$

$= \frac{\sqrt{3}}{2}$

$\Rightarrow$ width $= 1 + \sqrt{3}/2$

↳ working out LSF

$\frac{12}{1 + \sqrt{3}/2} = k$

$\therefore$ ASF $=$ LSF^2 $= \left(\frac{12}{1 + \sqrt{3}/2}\right)^2$

$\pi/2 \times \left(\frac{12}{1+\sqrt{3}/2}\right)$

$= 64.96\ldots m^2$

also know that 2 coats of paint needed

$64.96\ldots \times 2 = 129.92037\ldots$

$= 129.9m^2$ of paint

and for no. of tins $= \frac{129.9}{30} = 4.33\ldots$ tins

$= \boxed{\text{5 tins needed to buy}}$

4.

$$f(x)=\begin{cases}\dfrac{kx}{x^2+6} & \text{for } 0 \le x < 3 \\ \dfrac{k}{x^2-4} & \text{for } 3 \le x \quad \to x \geq 3\end{cases}$$

where k is a positive constant.

The area between the curve $y = f(x)$ and the positive x-axis is $\frac{1}{4}$

Show that

$$k = \frac{1}{\ln a}$$

where a is a constant to be determined.

(8)

(Total for Question 4 is 8 marks)

(a) focusing on the given ranges - see how actually we can rewrite area between curve $f(x)$ as two integrals - one between 3 and 0 and other between ∞ and 0

$$\overset{①}{\int_0^3 \frac{kx}{x^2+6}\,dx} + \overset{②}{\int_0^\infty \frac{kx}{x^2-4}\,dx}$$

for which the answer is given as 1/4

evaluate these separately

$$① \int_0^3 \frac{kx}{x^2+6}\,dx$$

} notice this is a fractional expression - numerator is a scalar multiple of the derivative of the denominator

∴ using reverse chain rule (step 2 from LHS list)

$$k\int \frac{f'(x)}{f(x)}\,dx = k\ln|f(x)|$$

consider: $\ln(x^2+6)$

differentiate: $\frac{1}{x^2+6} \times 2x = \frac{2x}{x^2+6}$
(chain rule)

...but need single x ∴

scale : $\times \frac{1}{2}$

$$\therefore \left[\frac{k}{2}\ln|x^2+6|\right]_0^3$$

subbing in

$$\left\{\left[\frac{k}{2}\ln((3)^2+6)\right] - \left[\frac{k}{2}\ln((0)^2+6)\right]\right\}$$

$$= \frac{k}{2}\ln(15) - \frac{k}{2}\ln(6)$$

factorise common $\frac{k}{2}$

$$\frac{k}{2}(\ln(15) - \ln(6))$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ *means* $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?
  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)}dx$
   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of -1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do
   $$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$$
   Method: copy $\ln$(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C$
   Recognisable by a power in the denominator other than
   $$\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3}\,dx \text{ etc}$$
4. **Is it Partial fractions!** Recognisable by products in the denominator.
   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$
   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$
   (only advanced courses have this form)
   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

and using log quotient law:

$$= \frac{k}{2}\ln\left(\frac{15}{6}\right)$$

next evaluate improper integral:

② $\int_0^{\infty} \frac{k}{x^2-4}\,dx$

first indefinitely $k\int \frac{1}{x^2-4}\,dx$

↳ notice this is a fractional expression where numerator is 1, now:

- can't split numerator as 1 in numerator
- CAN do partial fractions:

WAY 1 (easier!): noticing in formula book that in the form

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$$

$a^2 = 4$

$a = 2$

$$k\int \frac{1}{x^2-4} = \frac{k}{2(2)}\left[\ln\left(\frac{x-2}{x+2}\right)\right] + c$$

$$= \frac{k}{4}\left(\ln\left(\frac{x-2}{x+2}\right)\right) + c$$

WAY 2: doing integration of partial fractions

rewriting $k\int \frac{1}{x^2-4}\,dx$ as partial fractions

$$\frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\Rightarrow 1 = A(x-2) + B(x+2)$$

... comparing coefficients:

... x:

$0 = A + B$ -①

... constants:

$1 = -2A + 2B$ -②

solving simultaneously using equation solver or calc

①×2 -②

$$\begin{array}{r} 2A + 2B = 0 \\ - \quad -2A + 2B = 1 \\ \hline 4A = -1 \end{array}$$

$\div 4 \qquad \div 4$

$\Rightarrow A = -1/4$

subbing A into ①

... by substitution:

looking to make each bracket equal 0

$x = 2$,

$1 = 4B$

$\div 4 \qquad \div 4$

$B = 1/4$

$x = -2$,

$1 = A(-4)$

$\div -4 \qquad \div -4$

$\Rightarrow A = -1/4$

$0 = (-\tfrac{1}{4}) + B$

$\Rightarrow B = \tfrac{1}{4}$

$\therefore k\int \frac{1}{x^2-4} = k\int \left(\frac{-1}{4(x-2)} + \frac{1}{4(x-2)}\right) dx$

factorise ¼ out

$= \frac{k}{4}\int \left(\frac{1}{x-2} - \frac{1}{x+2}\right) dx$

integrate using fact that $\int \frac{1}{x} = \ln|x|$

$= \frac{k}{4}\left(\ln(x-2) - \ln(x+2)\right) + c$

now using log quotient law

$\frac{k}{4}\ln\left(\frac{x-2}{x+2}\right) + c$

now evaluate at limits

$k\int_3^\infty \frac{1}{x^2-4} dx = \lim_{t\to\infty} k\int_3^t \frac{1}{x^2-4} dx$

$= \lim_{t\to\infty} \frac{k}{4}\left[\ln\left(\frac{x-2}{x+2}\right)\right]_3^t$

$= \lim_{t\to\infty} \frac{k}{4}\left\{\left[\ln\left(\frac{t-2}{t+2}\right) - \ln\left(\frac{3-2}{3+2}\right)\right]\right.$

$= \lim_{t\to\infty} \frac{k}{4}\left[\ln\left(\frac{t-2}{t+2}\right) - \ln(\tfrac{1}{5})\right]$

now evaluate above at limits - using L'hospital rule for fractions (dividing numerator and denominator by 't' and evaluate at $t\to\infty$

$\ln\left(\frac{\frac{t-2}{t}}{\frac{t+2}{t}}\right) = \ln\left(\frac{1-\frac{2}{t}}{1+\frac{2}{t}}\right)$

as $t\to\infty$, $-\frac{2}{t}\to 0$, $\frac{2}{t}\to 0$

$\therefore \ln\left(\frac{t-2}{t+2}\right) \to \ln(1)$

$\therefore k\int_3^\infty \frac{1}{x^2-4} = \frac{k}{4}\left(\ln(1) - \ln(\tfrac{1}{5})\right)$, with $\ln(1) \to 0$

$= -\frac{k}{4}\ln(\tfrac{1}{5})$

subbing into initial strategy to exploit given ¼

$\frac{k}{2}\ln\left(\tfrac{15}{6}\right) + \left(-\frac{k}{4}\ln(\tfrac{1}{5})\right)$

$= \frac{2k}{4}\ln\left(\tfrac{15}{6}\right) - \frac{k}{4}\ln(\tfrac{1}{5})$

using log power rule for first expression

$$\frac{k}{4}\left(\ln\left(\tfrac{15}{6}\right)^2 - \ln\left(\tfrac{1}{5}\right)\right) = \tfrac{1}{4}$$

using log quotient rule

$$\frac{k}{\cancel{4}}\ln\left(\frac{\left(\tfrac{15}{6}\right)^2}{\tfrac{1}{5}}\right) = \tfrac{1}{\cancel{4}}$$

$$k\ln\left(\frac{125}{4}\right) = 1$$

$$\therefore k = \frac{1}{\ln\left(\tfrac{125}{4}\right)}$$

$$\boxed{\therefore a = \frac{125}{4}}$$

Year 1 Vectors - equations of planes, shortest distances, modelling with vectors

5.

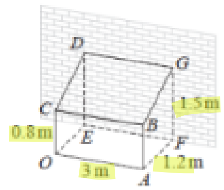


Figure 2

Figure 2 shows a sketch of a shelter against a wall. The shelter consists of two rectangular wooden boards, $OABC$ and $BCDG$, which can be modelled as parts of planes. Board $OABC$ is vertical and parallel to the wall and the ground may be assumed to be horizontal.
The points E and F are at the foot of the wall directly below D and G respectively.

The length OC is 0.8 m, the length OA is 3 m and the board $OABC$ is 1.2 m away from the wall. The points D and G are 1.5 m above the ground.

To model the shelter, take O as the origin, the vector $\mathbf{i}$ to be 1 m in the direction of $\overrightarrow{OA}$, the vector $\mathbf{j}$ to be 1 m in the direction of $\overrightarrow{OE}$ and the vector $\mathbf{k}$ to be 1 m in the direction of $\overrightarrow{OC}$.

(*a*) Find an equation of the plane $BCDG$, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$ **(5)**

In order to support the roof of the shelter, one end of a pole is attached to the ground at the centre of the rectangle $OAFE$ and the other end to a point on the roof. Modelling the pole as a rod,

(*b*) find, to the nearest mm, the shortest possible length for the pole. **(3)**

(*c*) State a limitation of the assumption that the boards can be modelled as planes. **(1)**

(Total for Question 5 is 9 marks)

(a) METHOD 1: vector parametric

to find scalar product equation for the plane BCDG - first need to find its vector parametric form i.e. $r = a + b\lambda + c\mu$

· to do this need to find 2 non parallel direction vectors on the plane - using given information and Fig 2

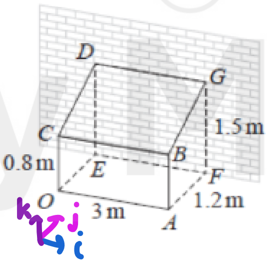


Figure 2

know that $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 0 \\ 0.8 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

and $\overrightarrow{DC} = \begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix} - \begin{pmatrix} 0 \\ 1.2 \\ 1.5 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ -1.2 \\ -0.7 \end{pmatrix} \overset{\text{-ve scalar multiple}}{=} \begin{pmatrix} 0 \\ 1.2 \\ 0.7 \end{pmatrix}$

and position vector can be C with $\begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix}$ ∴ subbing into general vector parametric formula

$$r = \begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1.2 \\ 0.7 \end{pmatrix}$$

now need to find the NORMAL to the plane BCDG ∴ a vector that is perpendicular to both direction vectors 'b' and 'c'

WAY 1: using dot product = 0

let vector = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

if perpendicular to both vectors, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$x(3) + y(0) + z(0) = 0$$

$$\Rightarrow 3x = 0$$

$$\div 3 \quad x = 0 \quad \div 3 \quad - ①$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1.2 \\ 0.7 \end{pmatrix} = 0$$

$$x(0) + y(1.2) + z(0.7) = 0$$

$$\Rightarrow 1.2y + 0.7z = 0 - ②$$

let $z = 1$,

$x = 0$ (from ①)

$1.2y + 0.7(1) = 0$ (from ②)

$\Rightarrow 1.2y = -0.7$

$\Rightarrow y = -7/12$

means we know x, y, z ∴ subbing into previously defined vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -7/12 \\ 1 \end{pmatrix}$$

∴ multiply by −12:

$$= \begin{pmatrix} 0 \\ 7 \\ -12 \end{pmatrix}$$

now take position vector 'a' and use $a \cdot n$ to get 'd'

$$\begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 7 \\ -12 \end{pmatrix} = 0(0) + 0(7) + 0.8(-12)$$

$$= -9.6$$

and subbing into scalar product for vectors, equation for BCDG is:

$$\boxed{r\cdot\begin{pmatrix}0\\7\\-12\end{pmatrix}=-9.6}$$

WAY 2 : vector cross product

$$\begin{pmatrix} i & j & k \\ 3 & 0 & 0 \\ 0 & 1.2 & 0.7\end{pmatrix} = i\begin{vmatrix}0 & 0\\1.2 & 0.7\end{vmatrix} - j\begin{vmatrix}3 & 0\\0 & 0.7\end{vmatrix} + k\begin{vmatrix}3 & 0\\0 & 1.2\end{vmatrix}$$

$$= \underline{i}(0) - \underline{j}(2.1) + (3.6)\underline{k}$$

$$\therefore \text{normal} = \begin{pmatrix}0\\-2.1\\3.6\end{pmatrix} \text{ or } \begin{pmatrix}0\\2.1\\-3.6\end{pmatrix}$$

now $a\cdot n = \underline{d}$

$$\underline{d} = \begin{pmatrix}0\\0\\0.8\end{pmatrix}\cdot\begin{pmatrix}0\\-2.1\\3.6\end{pmatrix} = 3.6(0.8) = 2.88$$

$$\Rightarrow \boxed{r\cdot\begin{pmatrix}0\\-2.1\\3.6\end{pmatrix} = 2.88}$$

(equivalent to way 1)

Alternative method : using Cartesian equation

first find vector parametric equation for BCDG – see Method 1 get

$$r = \begin{pmatrix}0\\0\\8\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1.2\\0.7\end{pmatrix}$$

now trying to get the normal and 'd' parts of $r\cdot n = d$ through getting BCDG's Cartesian equation – because 'r' is the general coordinate can rewrite it as $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0+3\lambda+0(\mu)\\0+\lambda(0)+\mu(1.2)\\0.8+\lambda(0)+\mu(0.7)\end{pmatrix}$$

Converting these into 3 linear equations

$$x = 3\lambda$$
$$y = 1.2\mu \quad -① $$
$$z = 0.8 + 0.7\mu \quad -②$$

and solving ① and ② simultaneously

subbing ① rearranged $\left(\mu = \frac{y}{1.2}\right)$ into ②

$$z = 0.8 + 0.7\left(\frac{y}{1.2}\right) \Rightarrow z = 0.8 + \frac{7}{12}y \Rightarrow 12z - 7y = 9.6$$

×12 ×12

Cartesian equtn of BCDG

because this is in the form $n_1x + n_2y + n_3z - d = 0$, can read normal as $\begin{pmatrix} 0 \\ -7 \\ 12 \end{pmatrix}$ and 'd' as 9.6

$$\boxed{\Pi : \mathbf{r} \cdot \begin{pmatrix} 0 \\ -7 \\ 12 \end{pmatrix} = 9.6}$$

(b) notice how the question is asking for the shortest distance between POINT (centre of rectangle) and plane

first finding coordinates of centre of rectangle :

see how only involves 'i' and 'j'

$$\therefore \begin{pmatrix} 0.5(3) \\ 0.5(1.2) \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.6 \\ 0 \end{pmatrix}$$

using formula booklet formula for shortest distance between point and plane:

perp. distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is:

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma - d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Subbing in above AND using 'n' and 'd' from part (a):

$\mathbf{r} \cdot \mathbf{n} = d$

$$\text{shortest distance} = \frac{|0(1.5) + (-7)(0.6) + 12(0) - 9.6|}{\sqrt{0^2 + (-7)^2 + (12)^2}}$$

$$= \frac{|-4.2 - 9.6|}{\sqrt{193}} = 0.993345...$$

$$\boxed{= 0.993 \ (3 s.f)}$$

(c) possible limitations:

- boards will not have negligible thickness which should be taken into account in the model
- wooden boards will bow and not form planes
- boards not completely that ∴ not suitable to model as planes

6.

$$f(x) = kx^2 + 3x - 11 \qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

where k and m are real constants.

Given that

- the sum of the roots of f is equal to the product of the roots of g
- g has at least one root on the imaginary axis

(*a*) solve completely

(i) $f(x) = 0$

(ii) $g(x) = 0$

(7)

(*b*) Plot the roots of f and the roots of g on a single Argand diagram.

(2)

(Total for Question 6 is 9 marks)

(a)

first finding 'sum of roots' of the quadratic $f(x)$

remembering how for a quadratic : ax^2+bx+c

sum of roots : $\Sigma\alpha = -b/a$

product of roots : $\alpha\beta = c/a$

$\therefore$ sum of roots $= -3/k$

now the product of roots of the cubic $g(x)$

...remembering how for a cubic : $ax^3+bx^2+cx+d=0$

sum of roots : $\Sigma\alpha = -b/a$

sum of product pairs : $\Sigma\alpha\beta = c/a$

product of roots : $\alpha\beta\gamma = -d/a$

$\therefore$ product of pairs is $\dfrac{-(-9)}{m} = 9/m$

equating

$-3/k = 9/m$ —①

next notice that g having at least one root on imaginary axis suggests that it has two roots that are a complex conjugate pair (according to the Fundamental Law of Algebra, if z is a root, so is z^*)

...2 ways to use above:

WAY 1 : using roots of polynomials

$\Sigma\alpha = -b/a$ = real root

$\therefore$ sum of roots $= \alpha+\beta+\gamma = 2/m$

but know sum of complex conjugate pairs of cubic $=0$

$$\Rightarrow \gamma = -\left(-\frac{2}{m}\right)$$

$$= \frac{2}{m} \text{ is a factor}$$

need value for 'm' to sub into ① AND find 'k' needed to get equation of cubic

$\therefore$ using FACTOR THEOREM - if $(x - 2/m)$ is a factor, then $g(2/m)=0$

$$m\left(\frac{2}{m}\right)^3 - 2\left(\frac{2}{m}\right)^2 + 3\left(\frac{2}{m}\right) - 9 = 0$$

$$\cancel{m}\left(\frac{8}{\cancel{m^3}\,m^2}\right) - \frac{8}{m^2} + \frac{6}{m} - 9 = 0$$

$$\cancel{\frac{8}{m^2}} - \cancel{\frac{8}{m^2}} + \frac{6}{m} - 9 = 0$$

$$\Rightarrow \frac{6}{m} = 9$$

$\times m$ $\quad$ $\times m$

$$6 = 9m$$

$\div 9$ $\quad$ $\div 9$

$$m = 6/9$$

$$= 2/3$$

WAY 2 : see that if on imaginary axis $\Rightarrow x = \alpha i$

hence using FACTOR THEOREM - if $(x - \alpha i)$ is a factor, $g(\alpha i) = 0$

$\therefore$ subbing into $g(x)$ and making $=0$

$\hookrightarrow g(\alpha i) = 0$

$$m(\alpha i)^3 - 2(\alpha i)^2 + 3(\alpha i) - 9 = 0$$

using $i^2 = -1$, $i^3 = -i$

$$-m(\alpha^3 i) - 2\alpha^2(-1) + 3\alpha i - 9 = 0$$

$$-m\alpha^3 i + 2\alpha^2 + 3\alpha i - 9 = 0$$

making real and imaginary components $= 0$

...real:

$$2\alpha^2 - 9 = 0$$

$$\Rightarrow 2\alpha^2 = 9$$

$\div 2$ $\quad$ $\div 2$

$$\alpha^2 = 9/2$$

...imaginary:

$$3\alpha - \alpha^3 m = 0$$

factorise α out: $\alpha(3 - \alpha^2 m) = 0$

$\Rightarrow 3-\alpha^2 m=0$

$\Rightarrow \alpha^2 m = 3$

$\div m \qquad \div m$

$\Rightarrow \alpha^2 = \frac{3}{m} \quad \therefore m = \frac{3}{\alpha^2}$

sub in $\alpha^2 = 9/2$

$\Rightarrow m = 3/\tfrac{9}{2} = 2/3$

Subbing $m = 2/3$ into ①

$$-\frac{3}{k} = \frac{9}{2/3}$$

cross multiply

$\Rightarrow 9k = -\cancel{3}\left(\frac{2}{\cancel{3}}\right)$

$9k = -2$

$\div 9 \qquad \div 9$

$k = -2/9$

↳ subbing into quadratic

$f(x) = -2/9\,x^2 + 3x - 11 = 0$

↳ solve on CALC equation solver or quadratic formula

$$x = \frac{-(-3 \pm \sqrt{(3)^2 - 4(-2/9)(-11)})}{2(-2/9)}$$

$$= \frac{3 \pm \sqrt{9 - \frac{88}{9}}}{-4/9}$$

using $i = \sqrt{-1}$

$$= \frac{3 \pm \sqrt{-i}\left(\sqrt{7/9}\right)}{4/9}$$

$$= \frac{3 \pm i\left(\sqrt{7}/3\right)}{4/9}$$

$$= \frac{3 \pm i\left(\sqrt{7}/3\right)}{4/9}$$

$\times 9 \qquad \times 9$

$$= \frac{9\left(3 \pm i\left(\sqrt{7}/3\right)\right)}{4}$$

$$= \frac{27 \pm i(3\sqrt{7})}{4}$$

$$x = \frac{27 \pm i(3\sqrt{7})}{4}$$

$$\boxed{\therefore 2 \text{ roots of } f(x) = \frac{27 \pm 3\sqrt{7}i}{4}}$$

(ii) for cubic, know the value of 'm' ∴ subbing into $g(x)$

$$g(x) = \frac{2}{3}x^3 - 2x^2 + 3x - 9 = 0$$

also know

$$\Sigma\alpha = \frac{2}{2/3} = 6/2 = 3 \Rightarrow 3 \text{ is a root}$$

...for other roots:

WAY 1: by inspection : unknown quadratic be ax^2+bx+c

$$(x-3)(ax^2+bx+c) = \frac{2}{3}x^3 - 2x^2 + 3x - 9$$

COMPARING coefficients:

...x^3:

$a = 2/3$

...x^2:

$-3a + b = -2$

sub in $a = 2/3$

$-\cancel{3}\left(\frac{2}{\cancel{3}}\right) + b = -2$

$-2 + b = -2$

$\Rightarrow b = 0$

...constants:

$-3c = -9$

$\div -3 \quad \div -3$

$\Rightarrow c = 3$

$$\therefore (x-3)\left(\frac{2}{3}x^2 + 3\right) = 0$$

or $\times 3$ in second bracket

$$(x-3)(2x^2+9) = 0$$

making each bracket equal 0

solve using calc equtn solver/quadratic solver

$$x = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(9)}}{2(2)}$$

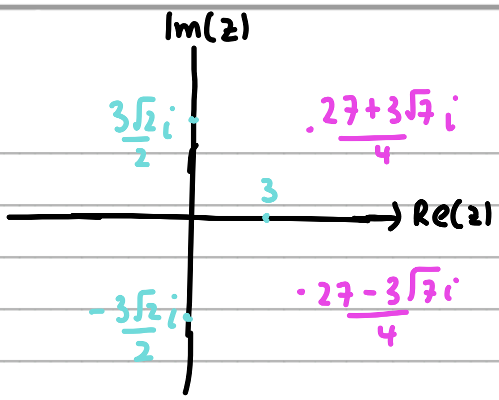
$$= \frac{\pm\sqrt{-72}}{4} = \frac{\pm\sqrt{-1}\,(\sqrt{72})}{4}$$

$$= \frac{\pm \cancel{6}^{3}\sqrt{2}\,i}{\cancel{4}_{2}} = \pm\frac{3\sqrt{2}}{2}i$$

∴ 3 roots of $g(x)$ are:

$$\boxed{3,\ \pm\frac{3\sqrt{2}}{2}i}$$

(b) plotting these roots on single Argand diagram: remembering complex numbers in form $a+bi$ are plotted as Cartesian coordinates (a, b)


Im(z)
Re(z)
3
$\frac{3\sqrt{2}}{2}i$
$-\frac{3\sqrt{2}}{2}i$
$\frac{27+3\sqrt{7}i}{4}$
$\frac{27-3\sqrt{7}i}{4}$

7. (i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \qquad \text{for all } n \in \mathbb{Z}^+$$

A student attempts to prove this statement using induction as follows.

Student's response

For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$

Suppose it is true for $n = k$, so $k^2 < 2^k$

Line 4 ⟶ Then $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)
$= 2k^2$
$< 2 \times 2^k$ (by the assumption $k^2 < 2^k$)
$= 2^{k+1}$

Hence the result is true for $n = k + 1$

So the result is true for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$, and hence it is true for all positive integers n by mathematical induction.

(*a*) Show by a counterexample that the statement is not true.

Given that the only mathematical error in the student's proof occurs in line 4,

(*b*) identify the error made in the student's proof,

(*c*) hence determine for which positive integers the statement is true, explaining your reasoning.

(5)

(Total for Question 7 is 11 marks)

(i) proving by induction means proving a conjecture is true for all $n \in \mathbb{N}$ - here are given a MATRIX proof by induction

Step 1: base case

prove statement is true for $n=1$

LHS:
$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^1$$

RHS:
$$\begin{pmatrix} 3(1)+1 & -(1) \\ 9(1) & 1-3(1) \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$

$\boxed{\text{LHS} = \text{RHS} \therefore \text{true for } n=1}$

Step 2: assumption step

assume statement is true for $n=k$

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^k = \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$$

step 3: induction step

prove statement is true for $n=k+1$

... AIM:

$$\begin{pmatrix} 3(k+1)+1 & -(k+1) \\ 9(k+1) & 1-3(k+1) \end{pmatrix} = \begin{pmatrix} 3k+4 & -k-1 \\ 9k+9 & -2-3k \end{pmatrix}$$

LHS:

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k+1} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^k \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$

subbing in matrix from assumption step:

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k+1} = \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$

↳ using matrix multiplication on RHS: multiply the elements in the row of the first matrix by the elements in the column of the second matrix and sum in between - REPEAT (see colours to indicate)

$$\begin{pmatrix} \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix} & \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix} \\ \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix} & \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (3k+1)(4)+(-k)(9) & (3k+1)(-1)+(-k)(-2) \\ 9k(4)+(1-3k)(9) & 9k(-1)+(1-3k)(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 12k+4-9k & -3k-1+2k \\ 36k+9-27k & -9k-2+6k \end{pmatrix} = \begin{pmatrix} 3k+4 & -k-1 \\ 9k+9 & -3k-2 \end{pmatrix} = \text{AIM} (\checkmark)$$

$\therefore$ true for $n=k+1$

step 4: conclusion step

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$

(ii)(a) trying for different 'n' values

$n=1 \Rightarrow (1)^2 < 2^{(1)}$
$1 < 2$ TRUE

$n=2 \Rightarrow (2)^2 < 2^{(2)}$
$4 < 4$ FALSE

$n=3 \Rightarrow (3)^2 < 2^3$
$9 < 8$ FALSE

$n=4 \Rightarrow (4)^2 < (2)^4$
$16 < 16$ FALSE

$n=5 \Rightarrow (5)^2 < (2)^5$
$25 < 32$ TRUE

$n=6 \Rightarrow (6)^2 < (2)^6$ $36 < 64$ TRUE

'false' are possible limitations //

(b) the statement $2k+1 < k^2$ is not true for all +ve integers $n \in \mathbb{Z}^+$

(c) the statement in line 4 is only true for +ve integers $k > 2$ so induction step only true for $n > 2 \Rightarrow$ induction holds from any base case greater than 2

- also see from (b) that result true for $k > 5$ $\therefore$ induction holds with base case $n = 5$
- but not true for 2, 3 or 4 (part (i))

$\boxed{\therefore \text{true for } k \in \mathbb{Z}^+ ;\ k \neq 2, 3, 4}$

8. A large container initially contains 3 litres of pure water.
Contaminated water starts pouring into the container at a constant rate of 250 ml per minute and you may assume the contaminant dissolves completely.

At the same time, the container is drained at a constant rate of 125 ml per minute.
The water in the container is continually mixed.

The amount of contaminant in the water pouring into the container, at time t minutes after pouring began, is modelled to be $(5 - e^{-0.1t})$ mg per litre.

Let m be the amount of contaminant, in milligrams, in the container at time t minutes after the contaminated water begins pouring into the container.

(*a*) (i) Write down an expression for the total volume of water in litres in the container at time t.

(ii) Hence show that the amount of contaminant in the container can be modelled by the differential equation

$$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24 + t}$$

(4)

(*b*) By solving the differential equation, find an expression for the amount of contaminant, in milligrams, in the container t minutes after the contaminated water begins to be poured into the container.

(8)

After 30 minutes, the concentration of contaminant in the water was measured as 3.79 mg per litre.

(*c*) Assess the model in light of this information, giving a reason for your answer.

(2)

(Total for Question 8 is 14 marks)

notice this is a modelling 1ODE 'filling the container' question ∴ need to follow the following format:

(a)(i) volume of liquid after 't' mins (litres): $3 + (0.25t - 0.125t)$

$3 + 0.125t$

(ii) amount of contaminant, in mg, in container at 't' mins: $\frac{m}{3 + 0.125t}$

amount of contaminant in: $0.25 \times 5 - e^{-0.1t}$

amount of contaminant out: $0.125 \times \frac{m}{3+0.125t}$

$\frac{dm}{dt}$ = rate in - rate out

$= 0.25(5 - e^{-0.1t}) - \frac{\cancel{0.125}^{1} m}{\cancel{3}^{24} + \cancel{0.125}^{1} t}$

$= \frac{1}{4}(5 - e^{-0.1t}) - \frac{m}{24 + t}$

(b) rearranging the given 1ODE to get in form:

$\frac{dy}{dx} + Py = Q \Rightarrow \frac{dm}{dt} + \frac{m}{24+t} = \frac{5 - e^{-0.1t}}{4}$

straight away can see that we cannot solve this 1ODE by separation of variables as it involves a subtraction, rather than the product of two variables and its derivatives $\frac{dy}{dx}$

next check for reverse product rule

$\frac{d}{dt}(m) = \frac{dm}{dt}$

$$\frac{dm}{dt} + \frac{m}{24+t} = \frac{5-e^{-0.1t}}{4}$$

$\frac{d}{dt}(1) \neq \frac{1}{24+t}$

hence need to introduce INTEGRATION FACTOR : $I.F = e^{\int P dt}$

$$= e^{\int \frac{1}{24+t} dt}$$

$$= e^{\ln(24+t)}$$

$$= 24+t$$

multiplying through by $24+t$

$$(24+t)\frac{dm}{dt} + m = \frac{5-e^{-0.1t}}{4}(24+t)$$

now checking for reverse product rule

↳ can rewrite LHS of the equation as the derivative of the product of m and $24+t$

$$\frac{d}{dt}\left((24+t)m\right) = \frac{5-e^{-0.1t}}{4}(24+t)$$

integrate both sides

$$(24+t)m = \int \frac{5-e^{-0.1t}}{4}(24+t)\, dt$$

expand the expression inside the RHS integral and taking the $1/4$ out in front

$$= \frac{1}{4}\int 120 + 5t - 24e^{-0.1t} - te^{-0.1t}\, dt$$

notice can integrate all using standard results EXCEPT the $te^{-0.1t}$ ∴ IBP

using **L**ogs
Inverse function
Algebraic expression
Trig functions

$$u = t \qquad v' = e^{-0.1t}$$
$$u' = 1 \qquad v = -10e^{-0.1t}$$

$$\int te^{-0.1t}\,dt = -10te^{-0.1t} - \int -10e^{-0.1t}\,dt$$
$$= -10te^{-0.1t} + 10\int e^{-0.1t}\,dt$$

using $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + c$

$$= -10te^{-0.1t} + 10\left[-\frac{1}{0.1} \times e^{-0.1t}\right] + c$$
$$= -10te^{-0.1t} - 100e^{-0.1t} + c$$

integrating rest of equation

$$(24+t)m = \frac{1}{4}\left[120t + \frac{5}{2}t^2 + 340e^{-0.1t} - \left(-10te^{-0.1t} - 100e^{-0.1t}\right)\right]$$

$\Rightarrow$ G.S : $(24+t)m = 30t + \frac{5}{8}t^2 + 85e^{-0.1t} + \frac{5}{2}te^{-0.1t} + c$

P.S : sub in initial conditions: at $t=0, m=0$

$$0 = 0 + 0 + 85 + 0 + c$$
$$0 = 85 + c$$
$$\Rightarrow c = -85$$

$$(24+t)m = 30t + 5/8\,t^2 + 85e^{-0.1t} + 5/2\,te^{-0.1t} - 85$$

$\div 24+t$

$$\boxed{m = \frac{1}{24+t}\left(30t + \frac{5}{8}t^2 + 85e^{-0.1t} + 5/2\,te^{-0.1} - 85\right)}$$

(c) Subbing $t = 30$ into part (b)

$$m = \frac{1}{24+(30)}\left(30(30) + \frac{5}{8}(30)^2 + 85e^{-0.1(30)} + \frac{5}{2}(30)e^{-0.1(30)} - 85\right)$$

$\Rightarrow m = 25.5567\ldots$

but need concentration using concentration $= \frac{\text{mass}}{\text{volume}}$

$$\text{concentration} = \frac{25.5567\ldots}{3 + 0.125(30)}$$
$$= 3.80100\ldots$$

which is close to actual value (to 1 d.p) AND can be explained by slight inaccuracies (suitable up to 30 mins)